Dynamics of a Coupled Atmosphere-Ocean Model

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Abstract

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We consider a coupled atmosphere-ocean model, which involves hydrodynamics, thermodynamics and nonautonomous interaction at the air-sea interface. First, we show that the coupled atmosphere-ocean system is stable under the external fluctuation in the atmospheric energy balance relation. Then, we estimate the atmospheric temperature feedback in terms of the freshwater flux, heat flux and the external fluctuation at the air-sea interface, as well as the earth's longwave radiation coefficient and the shortwave solar radiation profile. Finally, we prove that the coupled atmosphere-ocean system has time-periodic, quasiperiodic and almost periodic motions, whenever the external fluctuation in the atmospheric energy balance relation is time-periodic, quasiperiodic and almost periodic, respectively.

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Abbreviated Title: Coupled Atmosphere-Ocean Dynamics

1 Introduction: A coupled atmosphere-ocean model

The global ocean circulation consists of the wind-driven upper ocean circulation and a meridional overturning deep ocean circulation called the thermohaline circulation. The ocean thermohaline circulation involves water masses sinking at high latitudes and upwelling at lower latitudes. During the thermohaline circulation, water masses carry heat or cold around the globe. Thus, it is believed that the global ocean thermohaline circulation plays an important role in the climate [29].

The thermohaline circulation is maintained by water density contrasts in the ocean, which themselves are created by atmospheric forcing, namely, heat and freshwater exchange via evaporation and precipitation at the air-sea interface. Thus the thermohaline circulation is described by coupled atmosphere-ocean models [27, 29]. Such coupled models also describe feedback of the thermohaline circulation on the atmospheric dynamics (e.g., temperature feedback).

Mathematical models are a key component of our understanding of climate and geophysical systems. The formulation and analysis of mathematical models is central to the progress of better understanding of the thermohaline circulation dynamics and its impact on climate change.

We consider a coupled atmosphere-ocean model, with simplified atmospheric dynamics, i.e., the atmospheric dynamics is described by an energy balance model.

This is a zonally averaged, coupled atmosphere-ocean model on the meridional, latitude-depth (y, z)-plane as used by various authors [30, 35, 4, 8, 9, 12]. This model has been shown to capture some interesting climate phenomena [30, 35, 4]. It is composed of a one-dimensional stochastic energy balance model proposed by North and Cahalan [25], for the latitudinal atmosphere surface temperature $\theta(y, t)$, together with the Boussinesq equations for ocean dynamics in terms of stream function $\psi(y, z, t)$, and transport equations for the oceanic salinity $S(y, z, t)$ and the oceanic temperature $T(y, z, t)$ on the domain $D = \{(y, z) : 0 \leq y, z \leq 1\}$:

$$\begin{aligned}
\theta_t = \theta_{yy} - (a + \theta) \quad &+ \quad S_a(y) - \gamma(y)[S_o(y) + \theta - T] + f(y, t),\ 0 \leq y \leq 1, && (1.1)\\
q_t + J(q, \psi) \quad &= \quad Pr\Delta q + PrRa(T_y - S_y), \quad (y, z) \in D, && (1.2)\\
T_t + J(T, \psi) \quad &= \quad \Delta T, \quad (y, z) \in D, && (1.3)\\
S_t + J(S, \psi) \quad &= \quad \Delta S, \quad (y, z) \in D, && (1.4)
\end{aligned}$$

where $q(y, z, t) = -\Delta\psi$ is the vorticity; velocity field is $(v, w) = (\psi_z, -\psi_y)$; a is a positive constant parameterizing the effect of the earth's longwave radiative cooling; $S_a(y)$ and $S_o(y)$ are empirical functions representing the latitudinal dependence of the shortwave solar radiation; $\gamma(y)$ is the latitudinal fraction of the earth covered by the ocean basin; Pr

is the Prandtl number and Ra is the Rayleigh number. The first equation is the energy balance model proposed by North and Cahalan [25]. The forcing $f(y,t)$ may arise from, for example, eddy transport fluctuation, stormy bursts of latent heat, and flickering cloudiness variables. Moreover, $J(g,h) = g_x h_y - g_y h_x$ is the Jacobian operator and $\Delta = \partial_{yy} + \partial_{zz}$ is the Laplace operator. The effect of the rotation is parameterized in the magnitude of the viscosity and diffusivity terms as discussed in [33].
The no-flux boundary condition is taken for the atmosphere temperature $\theta(y,t)$

$$\theta_y(0,t) = \theta_y(1,t) = 0. \tag{1.5}$$

The fluid boundary condition is no normal flow and free-slip on the whole boundary

$$\psi = 0, \ q = 0. \tag{1.6}$$

The flux boundary conditions are assumed for the ocean temperature T and salinity S.
At top $z = 1$, the fluxes are specified as:

$$T_z = S_o(y) + (\theta - T)|_{z=1}, \quad S_z = F(y), \tag{1.7}$$

with $F(y)$ being the given freshwater flux.
At bottom $z = 0$:

$$T_z = S_z = 0\ . \tag{1.8}$$

On the lateral boundary $y = 0, 1$:

$$T_y = S_y = 0. \tag{1.9}$$

We also assume the following compatibility condition:

$$S_o'(0) = S_o'(1) = F'(0) = F'(1) = 0. \tag{1.10}$$

The non-autonomous partial differential equation (1.1) is only defined on the air-sea interface and it may be regarded as a dynamical boundary condition. The boundary condition (1.7) involves a coupling between the atmospheric and oceanic temperature at the air-sea interface.
In the next section, we discuss the well-posedness of this coupled atmosphere-ocean model. Then we investigate the stability of this coupled system under external fluctuation in §3, atmospheric temperature feedback in §4, time-periodic, quasiperiodic and almost periodic coupled motion in §5, respectively. Finally, we summarize these results in the final section §6.

2 Mathematical Setup

In order to use the standard result in [20] for the local existence, we homogenize inhomogeneous boundary conditions for T, S on the top boundary $z = 1$ as in [23].
First, we construct two scalar functions:

$$T_\epsilon^* = \tilde{T}^* \eta_\epsilon(z), \;\; S_\epsilon^* = \tilde{S}^* \eta_\epsilon(z), \; \forall \epsilon \in (0, \frac{1}{2}),$$

where

$$\begin{aligned} \tilde{T}^* &= [S_o(y) + \theta](1 - e^{1-z}), \\ \tilde{S}^* &= F(y)z, \\ \eta_\epsilon(z) &\in C^\infty([0,1]) \text{ is given by} \end{aligned}$$

$$\eta_\epsilon(z) = \begin{cases} 1, & 1 - \epsilon \le z \le 1, \\ \text{increasing}, & 1 - 2\epsilon \le z \le 1 - \epsilon, \\ 0, & 0 \le z \le 1 - 2\epsilon. \end{cases}$$

Then, set

$$\hat{T} = T - T_\epsilon^*, \;\; \hat{S} = S - S_\epsilon^*.$$

By (1.5) and (1.10), we see that the boundary conditions (1.7) for the new variables $\hat{T}$ and $\hat{S}$ are homogenized and do not affect other boundary conditions. Thus (1.1)–(1.9) become (for the simplicity, we still use T and S instead of $\hat{T}$ and $\hat{S}$)

$$\theta_t = \theta_{yy} - (a + \theta) + S_a(y) - \gamma(y)[S_o(y) + \theta - T] + f(y,t), \; 0 \le y \le 1, \tag{2.11}$$

$$q_t + J(q,\psi) = Pr\Delta q + PrRa(T_y - S_y + T_{\epsilon y}^* - S_{\epsilon y}^*), \quad (y,z) \in D, \tag{2.12}$$

$$\begin{aligned} T_t + J(T,\psi) + J(T_\epsilon^*,\psi) = \Delta T + \\ [((1 - e^{1-z})(1 - \gamma(y)) - e^{1-z})\eta_\epsilon(z) + (1 - e^{1-z})\eta_\epsilon''(z) + 2e^{1-z}\eta_\epsilon'(z) - e^{1-z}\eta_\epsilon]\theta \\ - (1 - e^{1-z})\eta_\epsilon(z)\gamma(y)T(y,1) + g, \quad (y,z) \in D, \end{aligned} \tag{2.13}$$

$$\begin{aligned} S_t + J(S,\psi) + J(S_\epsilon^*,\psi) = \Delta S \\ + F''(y)z\eta_\epsilon(z) + F(y)(2\eta_\epsilon'(z) + z\eta_\epsilon''(z)), \quad (y,z) \in D, \end{aligned} \tag{2.14}$$

where

$$\begin{aligned} g(y,z,t) = -(1 - e^{1-z})\eta_\epsilon(z)\{-a + S_a(y) - \gamma(y)S_o(y) + f(y,t) - S_o''(y)\} \\ + [(1 - e^{1-z})\eta_\epsilon''(z) + 2e^{1-z}\eta_\epsilon'(z) - e^{1-z}\eta_\epsilon]S_o(y). \end{aligned}$$

The corresponding boundary conditions become:
On the whole boundary, the fluid flow satisfies

$$\psi = 0, \quad q = 0. \tag{2.15}$$

The boundary conditions for the atmosphere temperature $\theta(y,t)$ (defined only on the air-sea interface) are

$$\theta_y(0,t) = \theta_y(1,t) = 0. \tag{2.16}$$

The boundary conditions for the ocean temperature T and salinity S become
At top $z = 1$:

$$T_z + T|_{z=1} = 0; \quad S_z = 0. \tag{2.17}$$

At bottom $z = 0$:

$$T_z = S_z = 0\ . \tag{2.18}$$

At the lateral boundary $y = 0, 1$:

$$T_y = S_y = 0. \tag{2.19}$$

The appropriate initial data θ_0, q_0, T_0, S_0 are also assumed.
Using the theory in [20], we can obtain the following local existence theorem for problem (2.1)–(2.9) (that is (1.1)–(1.9)).

Theorem 2.1 *(***Local Well-Posedness***) Let $\theta_0 \in H^1(0,1)$, $q_0 \in H_0^1(D)$, $T_0, S_0 \in H^1(D)$, $f \in L^\infty(0,\infty; L^2(0,1))$. Assume that the physical data satisfy $\gamma(y) \in L^\infty(0,1)$, and $S_o(y), F(y) \in H^2(0,1)$ and aslo assume that the compatibility condition (1.10) be satisfied. Then the coupled atmosphere-ocean system (2.11)–(2.19) (that is (1.1)–(1.9)) has a unique (The uniqueness of S is up to a constant) local solution satisfying*

$$\theta \in L^\infty(0,\tau; H^1(0,1)) \cap L^2(0,\tau; H^2(0,1)),$$

$$q \in L^\infty(0,\tau; H_0^1(D)) \cap L^2(0,\tau; H^2(D) \cap H_0^1(D)),$$

$$T, S \in L^\infty(0,\tau; H^1(D) \times H^1(D)) \cap L^2(0,\tau; H^2(D) \times H^2(D)),$$

where τ depends on initial data and physical data $S_a(y), S_o(y), F(y)$ and $f(y,t)$.

Since $-\Delta\psi = q \in H_0^1$, we get $\psi \in H_0^1(D) \cap H^3(D)$. Hence the Jacobian $J(\cdot,\cdot)$ is continuous from $H^1(D) \times H^3(D) \to L^2(D) \times L^2(D)$.

In order to obtain the global existence, we need a priori estimates. First, we give a priori estimates for (1.1)–(1.9) in L^2. In the sequel, $\|\cdot\|$ and $\|\cdot\|_1$ denote the norm of L^2 and H^1 respectively.

Multiplying (1.1) by θ and performing the integration by parts, we conclude that

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2 = -\|\theta_y\|^2 - \|\theta\|^2$$

$$-a\int_0^1 \theta dy + \int_0^1 S_a\theta dy - \int_0^1 r(y)[S_o + \theta - T]\theta dy + \int_0^1 f\theta dy, \tag{2.20}$$

By the Cauchy-Schwarz inequality, we arrive at

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2 \le -\|\theta_y\|^2 + C_1\|\theta\|^2 + \epsilon_1\|T(y,1)\|^2 + M_1, \tag{2.21}$$

where constant M_1 depends on $\|S_a||, \|S_o\|, a$ and $\sup_{0\le t<\infty}\|f\|$, constant C_1 depends on $\|\gamma\|_{L^\infty}$ and $\epsilon_1 > 0$, $\epsilon_1 > 0$ will be chosen later.

Multiplying (1.2) by q, performing the integration by parts and using the property of Jacobian and (1.6), we have

$$\frac{1}{2}\frac{d}{dt}\|q\|^2 = -Pr\|\nabla q\|^2 + PrRa\int_D (T_y - S_y)q. \tag{2.22}$$

Similarly, from (2.13) and (2.14), we have

$$\frac{1}{2}\frac{d}{dt}\|T\|^2 = -\|\nabla T\|^2 + \int_0^1 [S_o + \theta - T(y,1)]T(y,1)dy, \tag{2.23}$$

and

$$\frac{1}{2}\frac{d}{dt}\|S\|^2 = -\|\nabla S\|^2 + \int_0^1 F(y)S(y,1)dy. \tag{2.24}$$

Note that

$$\begin{aligned} PrRa\int_\Omega (T_y - S_y)q &\le \frac{Pr}{2\lambda_1}\|q\|^2 + \frac{PrRa^2\lambda_1}{2}(\|T_y\|^2 + \|S_y\|^2) \\ &\le \frac{Pr}{2}\|\nabla q\|^2 + PrRa^2\lambda_1(\|\nabla T\|^2 + \|\nabla S\|^2), \end{aligned}$$

where λ_1 is a constant in the inequality $\|v\|^2 \le \lambda_1\|\nabla v\|, v \in H_0^1$. Thus (2.22) can be rewritten as

$$\frac{1}{2}\frac{d}{dt}\|q\|^2 \le -\frac{Pr}{2}\|\nabla q\|^2 + PrRa^2\lambda_1(\|\nabla T\|^2 + \|\nabla S\|^2). \tag{2.25}$$

Multiplying (2.23) by $2PrRa^2\lambda_1$ and (2.24) by $2PrRa^2\lambda_1$, and adding to (2.25), we get

$$\frac{1}{2}\frac{d}{dt}(\|q\|^2 + 2PrRa^2\lambda_1(\|T\|^2 + \|S\|^2))$$

$$\le -\frac{Pr}{2}\|\nabla q\|^2 - PrRa^2\lambda_1(\|\nabla T\|^2 + \|\nabla S\|^2)$$

$$+ 2PrRa^2\lambda_1\int_0^1 [S_o + \theta - T(y,1)]T(y,1)dy + 2PrRa^2\lambda_1\int_0^1 F(y)S(y,1)dy. \tag{2.26}$$

By the Cauchy-Schwarz inequality and the trace inequality([13]), we have

$$\frac{1}{2}\frac{d}{dt}(\|q\|^2 + 2PrRa^2\lambda_1\|T\|^2 + 2PrRa^2\lambda_1\|S\|^2)$$

$$\leq -\frac{Pr}{2}\|\nabla q\|^2 - PrRa^2\lambda_1(\|\nabla T\|^2 + \|\nabla S\|^2)$$

$$+ C_2\|\theta\|^2 - \frac{1}{2}\|T(y,1)\|^2 + \epsilon_2(\|\nabla S\|^2 + \|S\|^2) + M_2, \tag{2.27}$$

where C_2 depends on Pr, Ra and λ_1, and M_2 depends on $Pr, Ra, \lambda_1, \|S_o\|$ and $\|F\|$. Choosing $\epsilon_1 < \frac{1}{2}$ and $\epsilon_2 < \frac{PrRa^2\lambda_1}{2}$, combining (2.21) with (2.27), we obtain

$$\frac{1}{2}\frac{d}{dt}(\|\theta\|^2 + \|q\|^2 + 2PrRa^2\lambda_1\|T\|^2 + 2PrRa^2\lambda_1\|S\|^2)$$

$$\leq -\alpha(\|\nabla\theta\|^2 + \|\nabla q\|^2 + \|\nabla T\|^2 + \|\nabla S\|^2) + C_3(\|\theta\|^2 + \|S\|^2) + M_3, \tag{2.28}$$

where C_3 depends on C_1 and C_2, M_3 depends on M_1 and M_2, and α is a positive constant depending on Pr, Ra and λ_1. By the Gronwall inequality, we have

$$\|\theta\|^2 + \|q\|^2 + \|T\|^2 + \|S\|^2$$

$$+ \int_0^b (\|\nabla\theta\|^2 + \|\nabla q\|^2 + \|\nabla T\|^2 + \|\nabla S\|^2)dt \leq C_1(b), \tag{2.29}$$

for any given future time $b(0 < b < \infty, 0 < t \leq b)$ and some positive constant $C_1(b)$ depending on b, C_3 and M_3. By a similar argument in [15] (here we need to obtain the estimates in $H^1 \times H^1 \times H^1 \times H^1 \times H^1$ for system of (2.11)-(2.14) in order to avoid the trouble of non-homogeneous boundary conditions, we omit the details here since we will give a similar proof in §4), we have

$$\|\nabla\theta\|^2 + \|\nabla q\|^2 + \|\nabla T\|^2 + \|\nabla S\|^2 +$$

$$\int_0^b (\|\Delta\theta\|^2 + \|\Delta q\|^2 + \|\Delta T\|^2 + \|\Delta S\|^2)dt \leq C_2(b), \tag{2.30}$$

for any given $b(0 < b < \infty)$ and some positive constant $C_2(b)$ depending on $\|S_o\|_{H^2}$, $\|F\|_{H^2}$, b and $C_1(b)$.

Remark 2.2 *In fact, the estimate we get in (2.30) is for $\hat{T}$ and $\hat{S}$, from which we can get the estimate for oringinal T and S using the estimate for θ.*

Remark 2.3 *Since the equivalence $\|T\|^2+\|\Delta T\|^2$ with $\|T\|^2_{H^2}$ and the same for S, together with (2.29), we can replace $\int_0^b(\|\Delta\theta\|^2+\|\Delta q\|^2+\|\Delta T\|^2+\|\Delta S\|^2)dt$ by $\int_0^b(\|\theta\|^2_{H^2}+\|q\|^2_{H^2}+\|T\|^2_{H^2}+\|S\|^2_{H^2})dt$ in the estimate (2.30).*

With these global estimates, we have the following global existence theorem for the coupled atmosphere-ocean system:

Theorem 2.4 *(***Global Well-Posedness***) Let $\theta_0 \in H^1(0,1)$, $q_0 \in H_0^1(D)$, $T_0, S_0 \in H^1(D)$, $f \in L^\infty(0,\infty; L^2(0,1))$. Assume that the physical data satisfy $\gamma(y) \in L^\infty(0,1)$, and $S_o(y), F(y) \in H^2(0,1)$ and aslo assume that the compatibility condition (1.10) be satisfied. Then for any given $b(0 < b < \infty)$, the coupled atmosphere-ocean system (2.11)–(2.19) (that is (1.1)–(1.9)) has a unique (The uniqueness of S is up to a constant) global solution satisfying*

$$\theta \in L^\infty(0,b; H^1(0,1)) \cap L^2(0,b; H^2(0,1)),$$

$$q \in L^\infty(0,b; H_0^1(D)) \cap L^2(0,b; H^2(D) \cap H_0^1(D)),$$

$$T, S \in L^\infty(0,b; H^1(D) \times H^1(D)) \cap L^2(0,b; H^2(D) \times H^2(D)).$$

In the rest of this paper, we assume the conditions in this theorem are satisfied, so that we always have global unique solutions.

In the next section, we consider the stability of the above coupled atmosphere-ocean system with respect to the external fluctuation $f(y,t)$ in the atmospheric energy balance dynamics (1.1).

3 Stability under External Fluctuation

Paleo-evidence on the instability of the thermohaline circulation is now abundant. Numerical work suggested that a sufficiently large external forcing (such as external fluctuations in the atmospheric energy balance model and the freshwater flux at the air-sea interface) could destabilize or shutdown the thermohaline circulation [28]. This indicates that current capacity of carrying heat poleward by the thermohaline circulation may change when the freshwater budget is altered. Since the thermohaline circulation's important role in redistributing the heat around the globe, a breakdown or instability of the current thermohaline circulation may lead to dramatic climate change [29]. Because of this relation between the thermohaline circulation and climate change, there is growing interest in its stability or instability. This makes the stability issue of the thermohaline circulation not only of scientific but also of great practical importance.

In this section, we prove the stability of the coupled atmosphere-ocean system with respect to the external fluctuation $f(y,t)$ in the atmospheric energy balance dynamics (1.1), i.e., the continuous dependence of solution on $f(y,t)$ in the space H^1.

Assume that $\{\theta_1, q_1, T_1, S_1\}$ and $\{\theta_2, q_2, T_2, S_2\}$ are solutions with respect to $f_1(y,t)$ and $f_2(y,t)$. Let

$$\bar{\theta} = \theta_1 - \theta_2,\ \bar{q} = q_1 - q_2,\ \bar{T} = T_1 - T_2,\ \bar{S} = S_1 - S_2,\ \bar{f} = f_1 - f_2,$$

then $\bar{\theta}, \bar{q}, \bar{T}$ and $\bar{S}$ satisfy

$$\bar{\theta}_t = \bar{\theta}_{yy} - \bar{\theta} \quad - \quad \gamma(y)[\bar{\theta} - \bar{T}] + \bar{f}(y,t), \qquad 0 \le z \le 1, \tag{3.1}$$

$$\bar{q}_t + J(q_1,\psi_1) - J(q_2,\psi_2) \quad = \quad Pr\Delta\bar{q} + PrRa(\bar{T}_y - \bar{S}_y)\ ,\ (y,z) \in D, \tag{3.2}$$

$$\bar{T}_t + J(T_1,\psi_1) - J(T_2,\psi_2) \quad = \quad \Delta\bar{T}\ ,\ \ (y,z) \in D, \tag{3.3}$$

$$\bar{S}_t + J(S_1,\psi_1) - J(S_2,\psi_2) \quad = \quad \Delta\bar{S}\ ,\ \ (y,z) \in D, \tag{3.4}$$

The corresponding boundary conditions are as follows.
On the whole boundary:

$$\bar{\psi} = 0,\ \bar{\Delta}\psi = \bar{q} = 0. \tag{3.5}$$

$$\bar{\theta}_y(0,t) = \bar{\theta}_y(1,t) = 0. \tag{3.6}$$

At top $z = 1$:

$$\bar{T}_z + \bar{T}|_{z=1} = \bar{\theta};\ \bar{S}_z = 0. \tag{3.7}$$

At bottom $z = 0$:

$$\bar{T}_z = \bar{S}_z = 0\ . \tag{3.8}$$

At the lateral boundary $y = 0, 1$:

$$\bar{T}_y = \bar{S}_y = 0. \tag{3.9}$$

Similar to the discussion in §2 above, we have

$$\frac{1}{2}\frac{d}{dt}\|\bar{\theta}\|^2 = -\|\bar{\theta}_y\|^2 - \|\bar{\theta}\|^2$$

$$-\int_0^1 r(y)|\bar{\theta}|^2 dy + \int_0^1 \gamma(y)T(y,1)\bar{\theta}dy + \int_0^1 \bar{f}\bar{\theta}dy, \tag{3.10}$$

$$\frac{1}{2}\frac{d}{dt}\|\bar{q}\|^2 + \int_D (J(q_1,\psi_1) - J(q_2,\psi_2))\bar{q} = -Pr\|\nabla\bar{q}\|^2 + PrRa\int_\Omega (\bar{T}_y - \bar{S}_y)\bar{q}, \tag{3.11}$$

$$\frac{1}{2}\frac{d}{dt}\|\bar{T}\|^2 + \int_D (J(T_1,\psi_1) - J(T_2,\psi_2))\bar{T} = -\|\nabla\bar{T}\|^2 + \int_0^1 [\bar{\theta} - \bar{T}(y,1)]\bar{T}(y,1)dy, \tag{3.12}$$

$$\frac{1}{2}\frac{d}{dt}\|\bar{S}\|^2 + \int_D (J(S_1,\psi_1) - J(S_2,\psi_2))\bar{S} = -\|\nabla\bar{S}\|^2. \tag{3.13}$$

In order to estimate the terms $\int_D (J(q_1,\psi_1) - J(q_2,\psi_2))\bar{q}$, $\int_D (J(T_1,\psi_1) - J(T_2,\psi_2))\bar{T}$ and $\int_D (J(S_1,\psi_1) - J(S_2,\psi_2))\bar{S}$, we need the following lemma:

Lemma 3.1 *The nonlinear Jacobian operaror $J(u,v)$ has the following property*

$$\|J(u_1,u_2)-J(v_1,v_2))\| \le$$
$$(\|\nabla u_1\|+\|\nabla u_2\|+\|\nabla v_1\|+\|\nabla v_2\|)(\|\nabla(u_1-v_1)\|+\|\nabla(u_2-v_2)\|), \tag{3.14}$$

for every $u_i, v_i \in H^1(D)$ $(i = 1,\ 2)$.

The proof of this lemma is in [16].
By **Lemma 3.1**, we have

$$\int_D (J(q_1,\psi_1)-J(q_2,\psi_2))\bar{q} \le (1+\lambda_1)^2(\|\nabla q_1\|+\|\nabla q_2\|)\|\nabla\bar{q}\|\|\bar{q}\|,$$

$$\int_D (J(T_1,\psi_1)-J(T_2,\psi_2))\bar{T}$$
$$\le (\|\nabla T_1\|+\|\nabla T_2\|+\lambda_1(\|\nabla q_1\|+\|\nabla q_2\|))(\|\nabla\bar{T}\|+\lambda_1\|\nabla\bar{q}\|)\|\bar{T}\|,$$

and

$$\int_D (J(S_1,\psi_1)-J(S_2,\psi_2))\bar{S}$$
$$\le (\|\nabla S_1\|+\|\nabla S_2\|+\lambda_1(\|\nabla q_1\|+\|\nabla q_2\|))(\|\nabla\bar{S}\|+\lambda_1\|\nabla\bar{q}\|)\|\bar{S}\|.$$

Note that

$$PrRa\int_\Omega(\bar{T}_y-\bar{S}_y)\bar{q} \le \frac{Pr}{2\lambda_1}\|\bar{q}\|^2+PrRa^2\lambda_1(\|\bar{T}_y\|^2+\|\bar{S}_y\|^2)$$
$$\le \frac{Pr}{2}\|\nabla\bar{q}\|^2+PrRa^2\lambda_1(\|\nabla\bar{T}\|^2+\|\nabla\bar{S}\|^2).$$

Then by a similar argument as in §2 and using the Cauchy-Schwarz inequality, we conclude that

$$\frac{1}{2}\frac{d}{dt}(\|\bar{\theta}\|^2+\|\bar{q}\|^2+2PrRa^2\lambda_1\|\bar{T}\|^2+2PrRa^2\lambda_1\|\bar{S}\|^2)$$
$$\le C_4(\|\bar{\theta}\|^2+\|\bar{q}\|^2+2PrRa^2\lambda_1\|\bar{T}\|^2+2PrRa^2\lambda_1\|\bar{S}\|^2)+\|\bar{f}\|^2, \tag{3.15}$$

where C_4 depends on $Pr, Ra,\ \lambda_1,\ \|\gamma\|_{L^\infty}$ as well as the H^1-norm of q, T and S. By the Gronwall inequality, we further have

$$\|\bar{\theta}\|^2+\|\bar{q}\|^2+\|\bar{T}\|^2+\|\bar{S}\|^2 \le C(b)\|\bar{f}\|^2, \tag{3.16}$$

for any given b $(0<b<\infty, 0<t\le b)$ and some positive constant $C(b)$ depending on b and C_4. Furthermore, we can obtain the similar estimates for the gradient of $\{\theta, q, T, S\}$, we omit the proof here, as the similar derivation will be done in §5. Thus the solution differences $\bar{\theta}, \bar{q}, \bar{T}, \bar{S}$ and $\bar{f}$ are bounded when the external fluctuation difference $\bar{f}$ is bounded. So we have the following stability theorem.

Theorem 3.2 *(***Stability under the external fluctuation***) The coupled atmosphere-ocean system (2.11)–(2.14) (that is (1.1)–(1.4)) is stable under the external fluctuation in the atmospheric energy balance model. Namely, the solution of the coupled system depend continuously on the external fluctuation f in H^1.*

4 Dissipativity and Atmospheric Temperature Feedback

The ocean and the atmosphere are constantly interacting through the air-sea exchange process. It is expected that the thermohaline circulation could provide feedback to the air temperature. This is a direct impact of the thermohaline circulation on the climate. It is desirable to predict or estimate this feedback.

To this end, let us estimate the air temperature θ in the mean-square norm, in terms of the freshwater flux $F(y)$, external fluctuation $f(y,t)$ in the energy balance model, the earth's longwave radiative cooling coefficient a, and the empirical functions $S_a(y)$ and $S_o(y)$ representing the latitudinal dependence of the shortwave solar radiation, as well as physical parameters Pr and Ra.

We will also show that the system generated by (1.1)–(1.9) is a dissipative system in the sense of [19] or [31] under some conditions, that is all solutions $\{\theta, q, T, S\}$ enter a bounded set (so-called *absorbing set*) in $H^1(0,1) \times H_0^1(D) \times H^1(D) \times H^1(D)$ after a finite time. Since

$$\frac{d}{dt}\int_\Omega S dydz = \int_0^1 F(y)dy = \text{constant}.$$

For simplicity, we assume that

$$\int_0^1 F(y)dy = 0, \quad \int_D S dydz = 0 \tag{4.1}$$

and

$$0 < \gamma(y) \leq 1 \text{ or } 0 \leq \gamma(y) < 1. \tag{4.2}$$

First, we derive a uniform estimate for $\{\theta, q, T, S\}$ in $L^2(0,1) \times L^2(D) \times L^2(D) \times L^2(D)$. Using the standard energy estimate as given in §2, we get

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2 = -\|\theta_y\|^2 - \|\theta\|^2$$

$$-a\int_0^1 \theta dy + \int_0^1 S_a\theta dy - \int_0^1 r(y)[S_o + \theta - T(y,1)]\theta dy + \int_0^1 f\theta dy$$
$$\leq -\|\theta_y\|^2 - (1-\epsilon)\|\theta\|^2 - \inf_{y\in[0,1]}\gamma(y)\|\theta\|^2$$
$$+\frac{1}{\epsilon}[a^2 + \|S_o\|^2 + \|S_a\|^2 + \|f\|^2] + \|r(y)\|_{L^\infty}\int_0^1 |\theta||T(y,1)|dy. \tag{4.3}$$

$$\frac{1}{2}\frac{d}{dt}(\|q\|^2 + 2PrRa^2\lambda_1(\|T\|^2 + \|S\|^2)) \leq$$
$$-\frac{Pr}{2}\|\nabla q\|^2 - PrPa^2\lambda_1(\|\nabla T\|^2 + \|\nabla S\|^2)$$

$$+2PrRa^2\lambda_1\int_0^1[S_o+\theta-T(y,1)]T(y,1)dy+2PrRa^2\lambda_1\int_0^1 F(y)S(y,1)dy.$$
$$\le -\frac{Pr}{2}\|\nabla q\|^2-PrPa^2\lambda_1(\|\nabla T\|^2+\|\nabla S\|^2)-2PrRa^2\lambda_1(1-\epsilon)\int_0^1|T(y,1)|^2dy$$
$$+2PrRa^2\lambda_1\int_0^1|\theta||T(y,1)|dy+\frac{PrRa^2\lambda_1}{\epsilon}\|S_o\|^2+\frac{Pr^2Ra^4\lambda_1^2}{\epsilon_1}\|F\|^2+\epsilon_1\|S(y,1)\|^2, \quad (4.4)$$

here $\epsilon>0$ and $\epsilon_1>0$ will be chosen later. By the trace inequality, we have

$$\|S(y,1)\|^2\le C(\|\nabla S\|^2+\|S\|^2)\le C(1+\bar\lambda_1)\|\nabla S\|^2,$$

where $\bar\lambda_1$ is the constant in the following Poincaré inequality (note that $\int_D Sdydz=0$)

$$\|S\|^2\le\bar\lambda_1\|\nabla S\|^2. \quad (4.5)$$

Choosing $\epsilon_1=\frac{PrRa^2\lambda_1}{2(1+\bar\lambda_1)C}$, then (4.4) can be written as

$$\frac{1}{2}\frac{d}{dt}(\|q\|^2+2PrRa^2\lambda_1(\|T\|^2+\|S\|^2))\le$$
$$-\frac{Pr}{2}\|\nabla q\|^2-PrPa^2\lambda_1(\|\nabla T\|^2+\frac{1}{2}\|\nabla S\|^2)-2PrRa^2\lambda_1(1-\epsilon)\int_0^1|T(y,1)|^2dy$$
$$+2PrRa^2\lambda_1\int_0^1|\theta||T(y,1)|dy+\frac{PrRa^2\lambda_1}{\epsilon}\|S_o\|^2+2PrRa^2\lambda_1(1+\bar\lambda_1)C\|F\|^2. \quad (4.6)$$

Then, multiplying (4.3) by $2PrRa^2\lambda_1$ and adding to (4.6), we get

$$\frac{1}{2}\frac{d}{dt}(\|q\|^2+2PrRa^2\lambda_1(\|\theta\|^2+\|T\|^2+\|S\|^2))\le$$
$$-2PrRa^2\lambda_1\|\theta_y\|^2-\frac{Pr}{2}\|\nabla q\|^2-PrPa^2\lambda_1(\|\nabla T\|^2+\frac{1}{2}\|\nabla S\|^2)$$
$$-2PrRa^2\lambda_1((1-\epsilon)\|\theta\|^2-\inf_{y\in[0,1]}\gamma(y)\|\theta\|^2-(1+\|\gamma\|_{L^\infty})\int_0^1|T(y,1)||\theta|dy+(1-\epsilon)\int_0^1|T(y,1)|^2dy)$$
$$+\frac{PrRa^2\lambda_1}{2\epsilon}[a^2+\frac{5}{4}\|S_o\|^2++\|S_a\|^2+\|f\|^2]+2PrRa^2\lambda_1(1+\bar\lambda_1)C\|F\|^2. \quad (4.7)$$

By (4.2), when $0\le\gamma<1$, we could choose ϵ such that (since $\inf_{y\in[0,1]}\gamma(y)=0$ now)

$$4(1-\epsilon)^2>(1+\|\gamma\|_{L^\infty})^2, \text{ i.e. } \epsilon<\frac{1-\|\gamma\|_{L^\infty}}{1+\|\gamma\|_{L^\infty}}:=\alpha_0.$$

For example, we choose $\epsilon=\frac{\alpha_0}{2}$, then

$$-2PrRa^2\lambda_1((1-\epsilon)\|\theta\|-(1+\|\gamma\|_{L^\infty})\int_0^1|T(y,1)||\theta|dy+(1-\epsilon)\int_0^1|T(y,1)|^2dy)$$

$$< -\frac{PrRa^2\lambda_1\alpha_0}{2}(\|\theta\|^2 + \|T(y,1)\|^2).$$

If $0 < \gamma(y) \le 1$ as in (4.2), we denote $\inf_{y\in[0,1]} \gamma(y) = \beta_0$. Then we take $\epsilon = \frac{\beta_0}{6}$ to obtain

$$-2PrRa^2\lambda_1((1-\epsilon+\beta_0)\|\theta\| - (1+\|\gamma\|_{L^\infty})\int_0^1 |T(y,1)||\theta|dy + (1-\epsilon)\int_0^1 |T(y,1)|^2 dy)$$

$$< -\frac{PrRa^2\lambda_1\beta_0}{6}(\|\theta\|^2 + \|T(y,1)\|^2).$$

So, in the case of $0 \le \gamma(y) < 1$, (4.7) can be written as

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|q\|^2 + 2PrRa^2\lambda_1(\|\theta\|^2 + \|T\|^2 + \|S\|^2)) \le \\
&-2PrRa^2\lambda_1\|\theta_y\|^2 - \frac{Pr}{2}\|\nabla q\|^2 - PrRa^2\lambda_1(\|\nabla T\|^2 + \frac{1}{2}\|\nabla S\|^2) \\
&-\frac{PrRa^2\lambda_1\alpha_0}{2}(\|\theta\|^2 + \|T(y,1)\|^2) \\
&+\frac{PrRa^2\lambda_1}{\alpha_0}[a^2 + \frac{5}{4}\|S_o\|^2 + \|S_a\|^2 + \|f\|^2] + 2PrRa^2\lambda_1(1+\bar{\lambda}_1)C\|F\|^2. \qquad (4.8)
\end{aligned}$$

For $0 < \gamma(y) < 1$, we will have similar estimate. Since

$$T^2(y,z) - T^2(y,1) = 2\int_1^z TT_z dz,$$

we further have

$$\|T\|^2 \le 2\int_0^1 |T(y,1)|^2 dy + 4\|\nabla T\|^2. \qquad (4.9)$$

Using the Poincaré inequlity again for q and letting $\alpha_1 = \min\{\frac{Pr\lambda_1}{4}, \frac{\alpha_0}{8}, \frac{1}{8}, \frac{\bar{\lambda}_1}{8}\}$ and $\beta = \min\{\frac{Pr\lambda_1}{4}, \frac{PrRa^2\lambda_1\alpha_0}{4}, \frac{PrRa^2\lambda_1}{4}\}$, the estimate (4.8) becomes

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|q\|^2 + 2PrRa^2\lambda_1(\|\theta\|^2 + \|T\|^2 + \|S\|^2)) \le \\
&-\alpha_1(\|q\|^2 + 2PrRa^2\lambda_1(\|\theta\|^2 + \|T\|^2 + \|S\|^2) \\
&-\beta_1(\|\theta_y\|^2 + \|\nabla q\|^2 + \|\nabla T\|^2 + \|\nabla S\|^2 + \|T(y,1)\|^2) \\
&+\frac{PrRa^2\lambda_1}{\alpha_0}[a^2 + \frac{5}{4}\|S_o\|^2 + \|f\|^2] + 2PrRa^2\lambda_1(1+\bar{\lambda}_1)C\|F\|^2. \qquad (4.10)
\end{aligned}$$

Using the Gronwall inequality, we finally obtain the mean-square norm estimate for the solution of the coupled atmosphere-ocean model:

$$\|q\|^2 + 2PrRa^2\lambda_1(\|\theta\|^2 + \|T\|^2 + \|S\|^2) \le$$

$$e^{-\alpha_1 t}(\|q_0\|^2 + 2PrRa^2\lambda_1(\|\theta_0\|^2 + \|T_0\|^2 + \|S_0\|^2))$$
$$+\frac{PrRa^2\lambda_1}{\alpha_1}[\frac{1}{\alpha_0}(a^2 + \frac{5}{4}\|S_o\|^2 + \|S_a\|^2 + \|f\|^2) + 2(1+\bar{\lambda}_1)C\|F\|^2] \tag{4.11}$$

In particular, we get the mean-square norm estimate for the atmospheric temperature feedback

$$\|\theta\|^2 \le e^{-\alpha_1 t}(\frac{1}{2PrRa^2\lambda_1}\|q_0\|^2 + \|\theta_0\|^2 + \|T_0\|^2 + \|S_0\|^2)$$
$$+\frac{1}{2\alpha_1}[\frac{1}{\alpha_0}(a^2 + \frac{5}{4}\|S_o\|^2 + \|S_a\|^2 + \|f\|^2) + 2(1+\bar{\lambda}_1)C\|F\|^2]. \tag{4.12}$$

This atmospheric temperature feedback estimate is in terms of physical quantities such as the freshwater flux $F(y)$, external fluctuation $f(y,t)$ in the energy balance model, the earth's longwave radiative cooling coefficient a, and the empirical functions $S_a(y)$ and $S_o(y)$ representing the latitudinal dependence of the shortwave solar radiation, as well as physical parameters Pr and Ra. Here λ_1 and $\bar{\lambda}_1$ are the constants in the Poincaré inequality on the domain D in the cases of zero Dirichlet boundary condition and zero mean value, respectively. Moreover, C is a constant depending only on the domain D, $\alpha_0 = \frac{1-\|\gamma\|_{L^\infty}}{1+\|\gamma\|_{L^\infty}}$, and $\alpha_1 = \min\{\frac{Pr\lambda_1}{4}, \frac{\alpha_0}{8}, \frac{1}{8}, \frac{\bar{\lambda}_1}{8}\}$.

We can furthermore derive solution estimate in H^1 norm. To do so, we first get from (4.10)

$$\int_t^{t+1}(\|\theta_y\|^2 + \|\nabla q\|^2 + \|\nabla T\|^2 + \|\nabla S\|^2 + \|T(y,1)\|^2) \le$$
$$\frac{1}{\beta_1}e^{-\alpha_0 t}(\|q_0\|^2 + 2PrRa^2\lambda_1(\|\theta_0\|^2 + \|T_0\|^2 + \|S_0\|^2))$$
$$+\frac{2PrRa^2\lambda_1}{\alpha_1\beta_1}(\frac{1}{\alpha_0}(a^2 + \frac{5}{4}\|S_o\|^2 + \|S_a\|^2 + \|f\|^2) + 2(1+\bar{\lambda}_1)C\|F\|^2). \tag{4.13}$$

So, let $\|q_0\|^2 + 2PrRa^2\lambda_1(\|\theta_0\|^2 + \|T_0\|^2 + \|S_0\|^2)$ be bounded by some (big) upper bound R^2 and denote $M^2 = \frac{PrRa^2\lambda_1}{\alpha_1}(\frac{1}{\alpha_0}(a^2 + \frac{5}{4}\|S_o\|^2 + \|S_a\|^2 + \|f\|^2) + 2(1+\bar{\lambda}_1)C\|F\|^2)$. Then there is a time $t^* \ge \frac{2}{\alpha_1}\ln\frac{R}{M}$ such that

$$\|q\|^2 + 2PrRa^2\lambda_1(\|\theta\|^2 + \|T\|^2 + \|S\|^2) \le 2M^2,\ t \ge t^* \tag{4.14}$$

and

$$\int_t^{t+1}(\|\theta_y\|^2 + \|\nabla q\|^2 + \|\nabla T\|^2 + \|\nabla S\|^2 + \|T(y,1)\|^2) \le \frac{3}{\beta_1}M^2,\ t \ge t^*. \tag{4.15}$$

Next, we derive a uniform estimate of gradient of $\{\theta, q, T, S\}$ in $L^2(0,1)\times L^2(D)\times L^2(D)\times L^2(D)$. In order to avoid the difficulty caused by the non-homogeneous boundary conditions, we use equations (2.11)–(2.19) instead of (1.1)–(1.9).

Multiplying (2.11)–(2.14) by $-\theta_{yy}, -\Delta q, -\Delta T$ and $-\Delta S$ respectively, integrating over $(0,1)$ and D, noting that $S^* \in H^2(D)$ is known and T^* is only dependent on θ, we get

$$\frac{1}{2}\frac{d}{dt}\|\theta_y\|^2 = -\|\theta_{yy}\|^2 - \|\theta_y\|^2$$

$$+ a\int_0^1 \theta_{yy}dy - \int_0^1 S_a\theta_{yy}dy + \int_0^1 r(y)[S_o + \theta - T(y,1)]\theta_{yy}dy - \int_0^1 f\theta_{yy}dy, \tag{4.16}$$

$$\frac{1}{2}\frac{d}{dt}\|\nabla q\|^2 = -Pr\|\Delta q\|^2 + \int_D J(q,\psi)\Delta q - \int_D PrRa(T_y - S_y + T_y^* - S_y^*)\Delta q, \tag{4.17}$$

$$\frac{1}{2}\frac{d}{dt}(\|\nabla T\|^2 + \|T(y,1)\|^2) = -\|\Delta T\|^2 + \int_D J(T+T^*,\psi)\Delta T$$

$$-\int_D[((1-e^{1-z})(1-\gamma(y)) - e^{1-z})\eta_\epsilon(z) + (1-e^{1-z})\eta_\epsilon''(z) + 2e^{1-z}\eta_\epsilon'(z) - e^{1-z}\eta_\epsilon]\theta\Delta T$$

$$+\int_D(1-e^{1-z})\eta_\epsilon(z)\gamma(y)T(y,1)\Delta T - \int_D g\Delta T, \tag{4.18}$$

$$\frac{1}{2}\frac{d}{dt}\|\nabla S\|^2 = -\|\Delta S\|^2 + \int_D J(S+S^*,\psi)\Delta S$$

$$-\int_D F''(y)z\eta_\epsilon(z)\Delta S - \int_D F(y)(2\eta_\epsilon' + z\eta_\epsilon''(z))\Delta S. \tag{4.19}$$

Note that

$$a\int_0^1 \theta_{yy}dy - \int_0^1 S_a\theta_{yy}dy + \int_0^1 r(y)[S_o + \theta - T(y,1)]\theta_{yy}dy - \int_0^1 f\theta_{yy}dy$$

$$\leq \frac{3\epsilon}{2}\|\theta_{yy}\|^2 + \frac{1}{4\epsilon}[a^2 + \|S_a\|^2 + \|S_o\|^2 + \|f\|^2 + \|\theta\|^2 + \|T(y,1)\|^2], \tag{4.20}$$

$$-\int_D PrRa(T_y - S_y + T_y^* - S_y^*)\Delta q \leq$$

$$\frac{Pr}{2}\|\Delta q\|^2 + \frac{5PrRa^2}{2}(\|\nabla T\|^2 + \|\nabla S\|^2 + \|\theta_y\|^2) + \frac{5PrRa^2}{2}(\|S_o'\|^2 + \|F'\|^2), \tag{4.21}$$

$$-\int_D[((1-e^{1-z})(1-\gamma(y)) - e^{1-z})\eta_\epsilon(z) + (1-e^{1-z})\eta_\epsilon''(z) + 2e^{1-z}\eta_\epsilon'(z) - e^{1-z}\eta_\epsilon]\theta\Delta T$$

$$+\int_D(1-e^{1-z})\eta_\epsilon(z)\gamma(y)T(y,1)\Delta T - \int_D g\Delta T$$

$$\leq \frac{1}{2}\|\Delta T\|^2 + C(a^2 + \|S_a\|^2 + \|S_o\|_{H^2}^2 + \|f\|^2 + \|\theta\|^2 + \|T(y,1)\|^2). \tag{4.22}$$

$$-\int_D F''(y)z\eta_\epsilon(z)\Delta S - \int_D F(y)(2\eta_\epsilon' + z\eta_\epsilon''(z))\Delta S \leq \frac{1}{2}\|\Delta S\|^2 + C\|F\|_{H^2}^2. \tag{4.23}$$

About the estimates of $\int_D J(q,\psi)\Delta q, \int_D J(T+T^*,\psi)\Delta T$ and $\int_D J(S+S^*,\psi)\Delta S$, similar to [15], we have

$$\int_D J(q,\psi)\Delta q \leq C\|\Delta\psi\|\|\nabla q\|\|\Delta q\| = C\|q\|\|\nabla q\|\|\Delta q\|, \tag{4.24}$$

$$\int_D J(T+T^*,\psi)\Delta T \le C\|q\|(\|\Delta T\|+\|\nabla T\|+\|T\|)\|\nabla T\|+\|q\|(\|\theta_y\|+\|S_o'\|)\|\Delta T\|, \quad (4.25)$$

$$\int_D J(S+S^*,\psi)\Delta S \le C\|q\|(\|\Delta S\|+\|\nabla S\|+\|S\|)\|\nabla S\|+\|q\|\|F'\|\|\Delta S\|. \quad (4.26)$$

Using the Cauchy-Schwarz inequality, (4.20)–(4.26) and (4.14), when $t \ge t^*$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\theta_y\|^2+\|\nabla q\|^2+3PrRa^2\lambda_1(\|\nabla T\|^2+\|T(y,1)\|^2+\|\nabla S\|^2)) \\
&\le C(\|\theta_y\|^2+\|\nabla q\|^2+3PrRa^2\lambda_1(\|\nabla T\|^2+\|T(y,1)\|^2+\|\nabla S\|^2) \\
&\qquad + C(a^2+\|S_a\|^2+\|S_o\|^2_{H^2}+\|f\|^2+\|F\|^2_{H^2}). \quad (4.27)
\end{aligned}$$

By (4.15) and a uniform Gronwall lemma ([31]), we obtain

$$\begin{aligned}
&\|\theta_y\|^2+\|\nabla q\|^2+3PrRa^2\lambda_1(\|\nabla T\|^2+\|T(y,1)\|^2+\|\nabla S\|^2) \\
&\qquad \le C(a^2+\|S_a\|^2+\|S_o\|^2_{H^2}+\|f\|^2+\|F\|^2_{H^2}). \quad (4.28)
\end{aligned}$$

By (4.14) and (4.28), we know there exists an absorbing sets $\mathcal{B}$ in $H^1(0,1)\times H_0^1(D)\times H^1(D)\times H^1(D)$ for the solution of (1.1)–(1.9):

$$\mathcal{B}=\{\{\theta,q,T,S\}:\ \|\theta\|_1^2+\|q\|_1^2+\|T\|_1^2+\|S\|_1^2 \le C(a^2+\|S_a\|^2+\|S_o\|^2_{H^2}+\|f\|^2+\|F\|^2_{H^2})\}, \quad (4.29)$$

that is, for every bounded set in $H^1(0,1)\times H^1(D)\times H^1(D)\times H^1(D)$, when $t \ge t^*+1$, the solution of (1.1)–(1.9) will enter into the $\mathcal{B}$.

We summarize our results in section in the following theorem.

Theorem 4.1 *(***Atmospheric temperature feedback and dissipativity***) Assume that the freshwater flux $F(y)$ has zero mean as in (4.1) and the ocean basin's latitudinal fraction function $\gamma(y)$ is bounded as in (4.2). Then the coupled atmosphere-ocean system (1.1)–(1.4) has an absorbing set in $H^1(0,1)\times H_0^1(D)\times H^1(D)\times H^1(D)$ as given in (4.29). More importantly, the atmospheric temperature feedback $\theta(y,t)$ is bounded in mean-square norm in terms of physical quantities such as the freshwater flux $F(y)$, external fluctuation $f(y,t)$ in the energy balance model, the earth's longwave radiative cooling coefficient a, and the empirical functions $S_a(y)$ and $S_o(y)$ representing the latitudinal dependence of the shortwave solar radiation, as well as the Prandtl number Pr and the Rayleigh number Ra as in (4.12).*

Remark 4.2 *Due to $\frac{d}{dt}\int_D \bar{S}=0$, and $\int_D \bar{S}_0=0$, we obtain $\int_D \bar{S}=0$. So, as seen in the discussion of this section, we know that the stability (proved in §3) under the exernal fluctuation is uniform in time t when $0<\gamma(y)\le 1$ or $0\le\gamma(y)<1$.*

5 Strong Contraction and Almost Periodic Atmosphere-Ocean Dynamics

In this section, we study the coupled atmosphere-ocean dynamical response to almost periodic (in particular, periodic and quasi-periodic) external fluctuation $f(y,t)$ in the atmospheric energy balance model (1.1). A central question is: Does the coupled atmosphere-ocean system respond almost periodically to almost periodic external fluctuation $f(y,t)$? To answer this question, we need to understand the strong contraction property of the coupled atmosphere-ocean system in the absorbing set $\mathcal{B}$ defined in (4.29). Let $\{\theta^i, q^i, T^i, S^i\}$ be two trajectories corresponding to initial values $\{\theta_0^i, q_0^i, T_0^i, S_0^i\} \in \mathcal{B}$ for $i = 1, 2$. Note that these trajectories remain inside $\mathcal{B}$ as B is a forward invariant set. Their difference

$$\delta\theta = \theta^1 - \theta^2,\ \delta q = q^1 - q^2,\ \delta T = T^1 - T^2,\ \delta S = S^1 - S^2$$

satisfy the following equations:

$$\begin{aligned}
\delta\theta_t = \delta\theta_{yy} - \delta\theta \quad &- \quad \gamma(y)[\delta\theta - \delta T], \qquad 0 \le z \le 1, &(5.1)\\
\delta q_t + J(q_1, \psi_1) - J(q_2, \psi_2) \quad &= \quad Pr\Delta\delta q + PrRa(\delta T_y - \delta S_y)\ ,\ (y,z) \in D, &(5.2)\\
\delta T_t + J(T_1, \psi_1) - J(T_2, \psi_2) \quad &= \quad \Delta\delta T\ ,\ (y,z) \in D, &(5.3)\\
\delta S_t + J(S_1, \psi_1) - J(S_2, \psi_2) \quad &= \quad \Delta\delta S\ ,\ (y,z) \in D, &(5.4)
\end{aligned}$$

The corresponding boundary conditions are:

$$\delta\theta_y(0,t) = \delta\theta_y(1,t) = 0. \tag{5.5}$$

On the whole boundary:

$$\delta\psi = 0,\ \delta q = 0. \tag{5.6}$$

At top $z = 1$:

$$\delta T_z + \delta T|_{z=1} = \delta\theta;\ \delta S_z = 0. \tag{5.7}$$

At bottom $z = 0$:

$$\delta T_z = \delta S_z = 0. \tag{5.8}$$

At the lateral boundary $y = 0, 1$:

$$\delta T_y = \delta S_y = 0. \tag{5.9}$$

The initial conditions are:

$$\delta\theta_0 = \theta_0^1 - \theta_0^2,\ \delta q_0 = q_0^1 - q_0^2,\ \delta T_0 = T_0^1 - T_0^2,\ \delta S_0 = S_0^1 - S_0^2, \tag{5.10}$$

where $\delta\theta_0 = \delta\theta(y,0), \delta q_0 = \delta q(y,z,0), \delta T_0 = \delta T(y,z,0)$ and $\delta S_0 = \delta S(y,z,0)$.

Using energy estimates as in §3, we have

$$\frac{1}{2}\frac{d}{dt}\|\delta\theta\|^2 = -\|\delta\theta_y\|^2 - \|\delta\theta\|^2 - \int_D \gamma(y)|\delta\theta|^2 dy - \int_D \gamma(y)\delta\theta\delta T(y,1)dy, \tag{5.11}$$

$$\frac{1}{2}\frac{d}{dt}\|\delta q\|^2 = -Pr\|\nabla\delta q\|^2 - \int_D (J(q^1,\psi^1) - J(q^2,\psi^2))\delta q - \int_D PrRa(\delta T_y - \delta S_y)\delta q, \tag{5.12}$$

$$\frac{1}{2}\frac{d}{dt}\|\delta T\|^2 = -\|\nabla\delta T\|^2$$

$$+\int_0^1 [\delta\theta - \delta T(y,1)]\delta T(y,1)dy - \int_D (J(T^1,\psi^1) - J(T^2,\psi^2))\delta T, \tag{5.13}$$

$$\frac{1}{2}\frac{d}{dt}\|\delta S\|^2 = -\|\nabla\delta S\|^2 - \int_D (J(S^1,\psi^1) - J(S^2,\psi^2))\delta S. \tag{5.14}$$

Using **Lemma 3.1**, we imply

$$-\int_D (J(q^1,\psi^1) - J(q^2,\psi^2))\delta q \le \|J(q^1,\psi^1) - J(q^2,\psi^2)\|\|\delta q\| \le$$

$$(\|\nabla q^1\| + \|\nabla q^2\| + \|\nabla\psi^1\| + \|\nabla\psi^2\|)(\|\nabla\delta q\| + \|\nabla\delta\psi\|)\|\delta q\|$$

$$\le (1+\lambda_1)^2\sqrt{\lambda_1}(\|q^1\| + \|q^2\|)\|\nabla\delta q\|^2,$$

$$-\int_D (J(T^1,\psi^1) - J(T^2,\psi^2))\delta T \le \|J(T^1,\psi^1) - J(T^2,\psi^2)\|\|\delta T\|$$

$$\le (\|\nabla T^1\| + \|\nabla T^2\| + \|\nabla\psi^1\| + \|\nabla\psi^2\|)(\|\nabla\delta T\| + \|\nabla\delta\psi\|)\|\delta T\|$$

$$\le (\|\nabla T^1\| + \|\nabla T^2\| + \lambda_1(\|\nabla q^1\| + \|\nabla q^2\|))((\|\nabla\delta T\| + \lambda_1\|\nabla\delta q\|)\|\delta T\|,$$

$$-\int_D (J(S^1,\psi^1) - J(S^2,\psi^2))\delta S \le \|J(S^1,\psi^1) - J(S^2,\psi^2)\|\|\delta S\|$$

$$\le (\|\nabla S^1\| + \|\nabla S^2\| + \|\nabla\psi^1\| + \|\nabla\psi^2\|)(\|\nabla\delta S\| + \|\nabla\delta\psi\|)\|\delta S\|$$

$$\le (\|\nabla S^1\| + \|\nabla S^2\| + \lambda_1(\|\nabla q^1\| + \|\nabla q^2\|))((\|\nabla\delta S\| + \lambda_1\|\nabla\delta q\|)\|\delta S\|.$$

Other terms in (5.11)–(5.14) can be estimated as in the proof of the existence of the absorbing set in the last section. So we have

$$\frac{1}{2}\frac{d}{dt}(\|\delta q\|^2 + 2PrRa^2\lambda_1(\|\delta\theta\|^2 + \|\delta T\|^2 + \|\delta S\|^2)) \le$$

$$-\alpha_1(\|\delta q\|^2 + 2PrRa^2\lambda_1(\|\delta\theta\|^2 + \|\delta T\|^2 + \|\delta S\|^2))$$

$$-\beta_1(\|\delta\theta_y\|^2 + \|\nabla\delta q\|^2 + \|\nabla\delta T\|^2 + \frac{1}{2}\|\nabla\delta S\|^2 + \|\delta T(y,1)\|^2)$$

$$+(1+\lambda_1)^2\sqrt{\lambda_1}(\|q^1\| + \|q^2\|)\|\nabla\delta q\|^2 +$$

$$2PrRa^2\lambda_1(\|\nabla T^1\| + \|\nabla T^2\| + \lambda_1(\|\nabla q^1\| + \|\nabla q^2\|))((\|\nabla\delta T\| + \lambda_1\|\nabla\delta q\|)\|\delta T\|$$

$$+2PrRa^2\lambda_1(\|\nabla S^1\| + \|\nabla S^2\| + \lambda_1(\|\nabla q^1\| + \|\nabla q^2\|))((\|\nabla\delta S\| + \lambda_1\|\nabla\delta q\|)\|\delta S\|.$$

Now we assume that $C(a^2 + \|S_a\|^2 + \|S_o\|^2_{H^2} + \|f\|^2 + \|F\|^2_{H^2})$ is small enough. This is a condition imposed on the physical quantities such as the freshwater flux $F(y)$, external fluctuation $f(y,t)$ in the energy balance model, the earth's longwave radiative cooling coefficient a, and the empirical functions $S_a(y)$ and $S_o(y)$ representing the latitudinal dependence of the shortwave solar radiation.

By the Cauchy-Schwarz inequality, (4.5), (4.9) and (4.29), we get

$$\frac{1}{2}\frac{d}{dt}(\|\delta q\|^2 + 2PrRa^2\lambda_1(\|\delta\theta\|^2 + \|\delta T\|^2 + \|\delta S\|^2)) \leq$$
$$-\alpha_1(\|\delta q\|^2 + 2PrRa^2\lambda_1(\|\delta\theta\|^2 + \|\delta T\|^2 + \|\delta S\|^2))$$
$$-\frac{\beta_1}{2}(\|\delta\theta_y\|^2 + \|\nabla\delta q\|^2 + \|\nabla\delta T\|^2 + \frac{1}{2}\|\nabla\delta S\|^2 + \|\delta T(y,1)\|^2). \tag{5.15}$$

By the Gronwall's inequality, we obtain the strong contraction in $L^2(0,1) \times L^2(D) \times L^2(D) \times L^2(D)$ for the solution of the coupled atmosphere-ocean system (1.1)–(1.9), that is

$$\|\delta q\|^2 + 2PrRa^2\lambda_1(\|\delta\theta\|^2 + \|\delta T\|^2 + \|\delta S\|^2) \leq$$
$$e^{-\alpha_1 t}(\|\delta q_0\|^2 + 2PrRa^2\lambda_1(\|\delta\theta_0\|^2 + \|\delta T_0\|^2 + \|\delta S_0\|^2)). \tag{5.16}$$

Next, we can show the strong contraction of gradient in $L^2(0,1) \times L^2(D) \times L^2(D) \times L^2(D)$. We also need to estimate it using (2.11)–(2.19). Noticing that S^* is independent of $\{\theta, q, T, S\}$ and T^* is only dependent on θ, we get

$$\delta\theta_t = \delta\theta_{yy} - \delta\theta - \gamma(y)[\delta\theta - \delta T(y,1)],\ 0 \leq z \leq 1, \tag{5.17}$$

$$\delta q_t = Pr\Delta\delta q$$
$$- J(q^1,\psi^1) + J(q^2,\psi^2) + PrRa(\delta T_y - \delta S_y) + \delta\theta(1 - e^{1-z})\eta_\epsilon(z)),\ (y,z) \in D, \tag{5.18}$$
$$\delta T_t = \Delta\delta T - J(T^1,\psi^1) + J(T^2,\psi^2)$$
$$-J((1-e^{1-z})\eta_\epsilon(z)\theta^1,\psi^1) + J((1-e^{1-z})\eta_\epsilon(z)\theta^1,\psi^2) - J((1-e^{1-z})\eta_\epsilon(z)S_o(y),\delta\psi)$$
$$[((1-e^{1-z})(1-\gamma(y)) - e^{1-z})\eta_\epsilon(z) + (1-e^{1-z})\eta''_\epsilon(z) + 2e^{1-z}\eta'_\epsilon(z) - e^{1-z}\eta_\epsilon]\delta\theta$$
$$-(1-e^{1-z})\eta_\epsilon(z)\gamma(y)\delta T(y,1),\ (y,z) \in D, \tag{5.19}$$

$$\delta S_t = \Delta\delta S - J(S^1,\psi^1) + J(S^2,\psi^2) + J(S^*_\epsilon,\delta\psi),\ (y,z) \in D. \tag{5.20}$$

The corresponding boundary conditions are:

$$\delta\theta_y(0,t) = \delta\theta_y(1,t) = 0. \tag{5.21}$$

On the whole boundary:

$$\delta\psi = 0,\ \delta\Delta\psi = \delta q = 0. \tag{5.22}$$

At top $z = 1$:

$$\delta T_z + \delta T|_{z=1} = 0; \quad \delta S_z = 0. \tag{5.23}$$

At bottom $z = 0$:

$$\delta T_z = \delta S_z = 0\ . \tag{5.24}$$

At the lateral boundary $y = 0, 1$:

$$\delta T_y = \delta S_y = 0. \tag{5.25}$$

The initial conditions are:

$$\delta\theta_0 = \theta_0^1 - \theta_0^2,\ \delta q_0 = q_0^1 - q_0^2,\ \delta T_0 = T_0^1 - T_0^2 + (1 - e^{1-z})\eta_\epsilon(z)\delta\theta_0,\ \delta S_0 = S_0^1 - S_0^2, \tag{5.26}$$

where $\delta\theta_0 = \delta\theta(y, 0), \delta q_0 = \delta q(y, z, 0), \delta T_0 = \delta T(y, z, 0)$ and $\delta S_0 = \delta S(y, z, 0)$.
Multiplying (5.17)–(5.20) by $-\delta\theta_{yy}, -\Delta\delta q, -\Delta\delta T$ and $-\Delta\delta S$ respectively, integrating over $(0, 1)$ and D, we get

$$\frac{1}{2}\frac{d}{dt}\|\delta\theta_y\|^2 = -\|\delta\theta_{yy}\|^2 - \|\delta\theta_y\|^2 + \int_0^1 r(y)[\delta\theta - \delta T(y, 1)]\delta\theta_{yy}dy, \tag{5.27}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\delta q\|^2 &= -Pr\|\Delta\delta q\|^2 + \int_D (J(q^1, \psi^1) - J(q^2, \psi^2))\Delta\delta q \\ &- PrRa\int_D (\delta T_y - \delta S_y)\Delta\delta q - \int_D (1 - e^{1-z})\eta_\epsilon(z))\delta\theta\Delta\delta q,\end{aligned} \tag{5.28}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\nabla\delta T\|^2 + \|\delta T(y, 1)\|^2) &= -\|\Delta\delta T\|^2 + \int_D (J(T^1, \psi^1) - J(T^2, \psi^2))\Delta\delta T \\ &+ \int_D (J((1 - e^{1-z})\eta_\epsilon(z)\theta^1, \psi^1) - J((1 - e^{1-z})\eta_\epsilon(z)\theta^2, \psi^2))\Delta\delta T \\ - \int_D [((1 - e^{1-z})(1 - \gamma(y)) - e^{1-z})\eta_\epsilon(z) &+ (1 - e^{1-z})\eta_\epsilon''(z) + 2e^{1-z}\eta_\epsilon'(z) - e^{1-z}\eta_\epsilon]\delta\theta\Delta\delta T \\ &+ \int_D (1 - e^{1-z})\eta_\epsilon(z)\gamma(y)\delta T(y, 1)\Delta\delta T \\ &+ \int_D J((1 - e^{1-z})\eta_\epsilon(z)S_o(y), \delta\psi)\Delta\delta T,\ (y, z) \in D,\end{aligned} \tag{5.29}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\delta S\|^2 &= -\|\Delta\delta S\|^2 \\ + \int_D (J(S^1, \psi^1) - J(S^2, \psi^2))\Delta\delta S &+ \int_D J(S_\epsilon^*, \delta\psi)\Delta\delta S,\ (y, z) \in D.\end{aligned} \tag{5.30}$$

Note that

$$\int_0^1 r(y)[\delta\theta - \delta T(y, 1)]\delta\theta_{yy}dy \leq \frac{1}{2}\|\delta\theta_{yy}\|^2 + \|\gamma(y)\|_{L^\infty}\|^2(\|\delta\theta\|^2 + \|\delta T(y, 1)\|^2),$$

$$-PrRa\int_D(\delta T_y-\delta S_y)\Delta\delta q\le\frac{Pr}{4}\|\Delta\delta q\|^2+2PrRa^2(\|\nabla\delta T\|^2+\|\nabla\delta S\|^2),$$

$$-\int_D(1-e^{1-z})\eta_\epsilon(z))\delta\theta\Delta\delta q\le\frac{Pr}{4}\|\Delta\delta q\|^2+\frac{1}{Pr}\|\delta\theta\|^2,$$

$$-\int_D[((1-e^{1-z})(1-\gamma(y))-e^{1-z})\eta_\epsilon(z)+(1-e^{1-z})\eta_\epsilon''(z)+2e^{1-z}\eta_\epsilon'(z)-e^{1-z}\eta_\epsilon]\delta\theta\Delta\delta T$$

$$\le\frac{1}{6}\|\Delta\delta T\|^2+C\|\delta\theta\|^2,$$

$$\int_D(1-e^{1-z})\eta_\epsilon(z)\gamma(y)\delta T(y,1)\Delta\delta T\le\frac{1}{6}\|\Delta\delta T\|^2+\frac{3}{2}\|\delta T(y,1)\|^2,$$

$$\int_D J((1-e^{1-z})\eta_\epsilon(z)S_o(y),\delta\psi)\Delta\delta T\le\frac{1}{6}\|\Delta\delta T\|^2+C\lambda_1(\|S_o\|^2+\|S_o'\|^2)\|q\|^2,$$

$$\int_D J(S_\epsilon^*,\delta\psi)\Delta\delta S\le\frac{1}{2}\|\Delta\delta S\|^2+C\lambda_1(\|F\|^2+\|F'\|^2)\|q\|^2.$$

Now by **Lemma 3.1**, we get

$$\int_D(J(q^1,\psi^1)-J(q^2,\psi^2))\Delta\delta q\le\|\Delta\delta q\|\|J(q^1,\psi^1)-J(q^2,\psi^2)\|$$

$$\le\frac{1}{2}\|\Delta\delta q\|^2+C(1+\lambda_1)^2\|\nabla\delta q\|^2,$$

$$\int_D(J(T^1,\psi^1)-J(T^2,\psi^2))\Delta\delta T\le\|\Delta\delta T\|\|J(T^1,\psi^1)-J(T^2,\psi^2)\|$$

$$\le\frac{1}{4}\|\Delta\delta T\|^2+C(\|\nabla\delta q\|^2+|\nabla\delta T\|^2),$$

$$\int_D(J((1-e^{1-z})\eta_\epsilon(z)\theta^1,\psi^1)-J((1-e^{1-z})\eta_\epsilon(z)\theta^2,\psi^2))\Delta\delta T$$

$$\le\frac{1}{4}\|\Delta\delta T\|^2+C(\|\nabla\delta q\|^2+\|\nabla\delta\theta\|^2),$$

$$\int_D(J(S^1,\psi^1)-J(S^2,\psi^2))\Delta\delta S\le\frac{1}{2}\|\Delta\delta S\|^2+C(\|\nabla\delta q\|^2+|\nabla\delta S\|^2).$$

Putting the above estimtates together, we conclude

$$\frac{1}{2}\frac{d}{dt}(\|\delta\theta_y\|^2+\|\nabla\delta q\|^2+\|\delta T(y,1)\|^2+\|\nabla\delta T\|^2+\|\nabla\delta S\|^2)$$

$$\le C(\|\delta\theta\|^2+\|\delta T(y,1)\|^2+\|\nabla\delta q\|^2+\|\nabla\delta T\|^2+\|\nabla\delta S\|^2).\qquad(5.31)$$

Taking N large enough, multiplying (5.15) by N and then adding to (5.31), we imply that there exists a positive constant α_2 such that

$$\frac{1}{2}\frac{d}{dt}(N\|\delta q\|^2+2NPrRa^2\lambda_1(\|\delta\theta\|^2+\|\delta T\|^2+\|\delta S\|^2))$$

$$
\begin{aligned}
&+\frac{1}{2}\frac{d}{dt}(\|\delta\theta_y\|^2+\|\nabla\delta q\|^2+\|\delta T(y,1)\|^2+\|\nabla\delta T\|^2+\|\nabla\delta S\|^2)\\
&\le -\alpha_2(N\|\delta q\|^2+2NPrRa^2\lambda_1(\|\delta\theta\|^2+\|\delta T\|^2+\|\delta S\|^2))\\
&-\alpha_2(\|\delta\theta_y\|^2+\|\nabla\delta q\|^2+\|\nabla\delta T\|^2+\frac{1}{2}\|\nabla\delta S\|^2+\|\delta T(y,1)\|^2). \qquad (5.32)
\end{aligned}
$$

By the Gronwall inequality, we have

$$
\begin{aligned}
&N\|\delta q\|^2+2NPrRa^2\lambda_1(\|\delta\theta\|^2+\|\delta T\|^2+\|\delta S\|^2)\\
&+\|\delta\theta_y\|^2+\|\nabla\delta q\|^2+\|\delta T(y,1)\|^2+\|\nabla\delta T\|^2+\|\nabla\delta S\|^2\\
&\le e^{-\alpha_2 t}(N\|\delta q_0\|^2+2NPrRa^2\lambda_1(\|\delta\theta_0\|^2+\|\delta T_0\|^2+\|\delta S_0\|^2)\\
&e^{-\alpha_2 t}(\|\delta\theta_{0y}\|^2+\|\nabla\delta q_0\|^2+\|\delta T_0(y,1)\|^2+\|\nabla\delta T_0\|^2+\|\nabla\delta S_0\|^2). \qquad (5.33)
\end{aligned}
$$

This estimate tells us that any two solution trajectories inside the absorbing set approach each other as time goes on. This is the so called strong contraction property.

Remark 5.1 *In fact, here we get the estimate is for $\|\nabla\widehat{\delta T}\|^2$ and $\|\nabla\widehat{\delta S}\|^2=\|\nabla\delta S\|^2$. But $\|\nabla\widehat{\delta T}\|^2=\|\nabla(\delta T+(1-e^{1-z})\eta_\epsilon(z)\delta\theta)\|^2$, $\|\nabla\widehat{\delta S}\|^2=\|\nabla\delta S\|^2$ and we have the estimate for $\|\delta\theta\|_1$. Hence we can obtain the similar estimate for the original δT.*

Therefore we have the following theorem.

Theorem 5.2 *(**Strong contraction property**) Assume that the freshwater flux $F(y)$ has zero mean as in (4.1) and the ocean basin's latitudinal fraction function $\gamma(y)$ is bounded as in (4.2). Let $a^2+\|S_a\|^2+\|S_o\|^2_{H^2}+\|f\|^2+\|F\|^2_{H^2}$ be small enough. Then the coupled atmosphere-ocean system (1.1)–(1.4) has the strong contraction property.*

Now we come back to the issue of the time-almost periodic (in particular, time-periodic and time-quasi-periodic) motion in the coupled atmosphere-ocean system. First, we give the definitions about almost periodic function and pullback attractor.

A function $\varphi:I\!R\to X$, where (X,d_X) is a metric space, is called *almost periodic* [1] and [32] if for every $\varepsilon>0$ there exists a relatively dense subset M_ε of $I\!R$ such that

$$d_X(\varphi(t+\tau),\varphi(t))<\varepsilon$$

for all $t\in I\!R$ and $\tau\in M_\varepsilon$. A subset $M\subseteq I\!R$ is called *relatively dense* in $I\!R$ if there exists a positive number $l\in I\!R$ such that for every $a\in I\!R$ the interval $[a,a+l]\bigcap I\!R$ of length l contains an element of M, i.e. $M\bigcap[a,a+l]\neq\emptyset$ for every $a\in I\!R$.

In order to study the temporally almost periodic solutions of (1.1)–(1.9), we need some results from the theory of nonautonomous dynamical systems. Consider first an autonomous dynamical system on a metric space P described by a group $\Theta = \{\theta_t\}_{t \in I\!R}$ of mappings of P into itself.

Let X be a complete metric space and consider a continuous mapping

$$\Phi : I\!R^+ \times P \times X \to X$$

satisfying the properties

$$\Phi(0, p, \cdot) = \mathrm{id}_X, \qquad \Phi(\tau + t, p, x) = \Phi(\tau, \theta_t p, \Phi(t, p, x))$$

for all t, $\tau \in I\!R^+$, $p \in P$ and $x \in X$. The mapping Φ is called a cocycle on X with respect to Θ on P.

The appropriate concept of an attractor for a nonautonomous cocycle systems is the *pullback attractor*. In contrast to autonomous attractors it consists of a family subsets of the original state space X that are indexed by the cocycle parameter set.

A family $\widehat{A} = \{A_p\}_{p \in P}$ of nonempty compact sets of X is called a pullback attractor of the cocycle Φ on X with respect to θ_t on P if it is Φ–invariant, i.e.

$$\Phi(t, p, A_p) = A_{\theta_t p} \qquad \text{for all} \quad t \in I\!R^+, p \in P,$$

and pullback attracting, i.e.

$$\lim_{t \to \infty} H_X^* \left(\Phi(t, \theta_{-t} p, D), A_p \right) = 0 \qquad \text{for all} \quad D \in K(X),\ p \in P,$$

where $K(X)$ is the space of all nonempty compact subsets of the metric space (X, d_X). Here H_X^* is the Hausdorff semi–metric between nonempty compact subsets of X, i.e. $H_X^*(A, B) := \max_{a \in A} \mathrm{dist}(a, B) = \max_{a \in A} \min_{b \in B} d_X(A, b)$ for A, $B \in K(X)$.

The following theorem combines several known results. See Crauel and Flandoli [7], Flandoli and Schmalfuß [14], and Cheban [2] as well as [22, 6, 3] for this and various related proofs.

Theorem 5.3 *Let Φ be a continuous cocycle on a metric space X with respect to a group Θ of continuous mappings on a metric space P. In addition, suppose that there is a nonempty compact subset B of X and that for every $D \in K(X)$ there exists a $T(D) \in I\!R^+$, which is independent of $p \in P$, such that*

$$\Phi(t, p, D) \subset B \quad \textit{for all} \quad t > T(D). \tag{5.34}$$

Then there exists a unique pullback attractor $\widehat{A} = \{A_p\}_{p \in P}$ of the cocycle Φ on X, where

$$A_p = \bigcap_{\tau \in I\!R^+} \overline{\bigcup_{\substack{t > \tau \\ t \in I\!R^+}} \Phi\left(t, \theta_{-t} p, B\right)}. \tag{5.35}$$

Moreover, if the cocycle Φ is strongly contracting inside the absorbing set B. Then the pullback attractor consists of singleton valued sets, i.e. $A_p = \{a^*(p)\}$, and the mapping $p \mapsto a^*(p)$ is continuous.

The solution operators S_{t,t_0} for (1.1)–(1.9) form a cocycle mapping on $X = H^1(0,1) \times H_0^1(D) \times H^1(D) \times H^1(D)$ with parameter set $P = I\!R$, where $p = t_0$, the initial time, and $\theta_t t_0 = t_0 + t$, the left shift by time t. However, the space $P = I\!R$ is not compact here. Though more complicated, it is more useful to consider P to be the closure of the subset $\{\theta_t f, t \in I\!R\}$, i.e. the hull of f, in the metric space $L^2_{loc}(I\!R, X))$ of locally $L^2(I\!R)$–functions $f : I\!R \to X$ with the metric

$$d_P(f,g) := \sum_{N=1}^{\infty} 2^{-N} \min \left\{ 1, \sqrt{\int_{-N}^{N} |||f(t) - g(t)|||^2 \, dt} \right\}$$

with θ_t defined to be the left shift operator, i.e. $\theta_t f(\cdot) := f(\cdot + t)$, where $||| \cdot |||$ denotes the norm in X. By a classical result [1, 32], a function f in the above metric space is almost periodic if and only if the the hull of f is compact and minimal. Here minimal means nonempty, closed and invariant with respect to the autonomous dynamical system generated by the shift operators θ_t such that with no proper subset has these properties. The cocycle mapping is defined to be the solution $\vec{\omega}(t) = \{\theta, q, T, S\}$ of (1.1)–(1.9) starting at $\vec{\omega}_0 = \{\theta_0, q_0, T_0, S_0\} \in X$ at time $t_0 = 0$ for a given forcing mapping $f \in P$, i.e.

$$\Phi(t, f, \omega_0) := S^f_{t,0}\, \omega_0,$$

where we have included a superscript f on S to denote the dependence on the forcing term f. (This dependence is in fact continuous, see §3). The cocycle property here follows from the fact that $S^f_{t,t_0} \vec{\omega}_0 = S^{\theta_{t_0} f}_{t-t_0,0}\, \vec{\omega}_0$ for all $t \geq t_0$, $t_0 \in I\!R$, $\vec{\omega}_0 \in X$ and $f \in P$.

Following Theorem 5.3 and the dissipativity and strong contraction properties shown in the last two sections, we conclude that the coupled atmosphere-ocean system (1.1)–(1.4) has a unique pullback attractor, consists of the singleton valued component $\{\vec{a}^*(p)\} \in \hat{A}$ and the mapping $p \mapsto \vec{a}^*(p)$ is continuous on P. As in Duan and Kloeden [10] or Gao, Duan and Fu [17], we now show that this singleton attractor $\vec{a}^*(p)$ defines an almost periodic solution.

In fact, the mapping $p \mapsto \vec{a}^*(p)$ is uniformly continuous on P because P is compact subset of $L^2_{loc}(I\!R, X))$ due to the assumed almost periodicity. That is, for every $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $\|\vec{a}^*(p) - \vec{a}^*(q)\| < \varepsilon$ whenever $d_P(p,q) < \delta$. Now let the point $\bar{p}$ ($= f$, the given temporal forcing function) be almost periodic and for $\delta = \delta(\varepsilon) > 0$ denote by M_δ the relatively dense subset of $I\!R$ such that $d_P(\theta_{t+\tau}\bar{p}, \theta_t \bar{p}) < \delta$ for all $\tau \in M_\delta$ and $t \in$

$I\!R$. From this and the uniform continuity we have

$$\|\vec{a}^*(\theta_{t+\tau}\bar{p}) - \vec{a}^*(\theta_t\bar{p})\| < \varepsilon$$

for all $t \in I\!R$ and $\tau \in M_{\delta(\varepsilon)}$. Hence $t \mapsto \{\theta^*, q^*, T^*, S^*\}(t) := \vec{a}^*(\theta_t\bar{p})$ is almost periodic, and it is a solution of the coupled atmosphere-ocean model. It is unique as the single-trajectory pullback attractor is the only trajectory that exists and is bounded for the entire time line. Therefore, we have the following result.

Theorem 5.4 *(***Periodic, quasiperiodic and almost periodic motion***) Assume that the freshwater flux $F(y)$ has zero mean as in (4.1) and the ocean basin's latitudinal fraction function $\gamma(y)$ is bounded as in (4.2). Let $a^2 + \|S_a\|^2 + \|S_o\|^2_{H^2} + \|f\|^2 + \|F\|^2_{H^2}$ be small enough. Then the coupled atmosphere-ocean system (1.1)–(1.4) has unique time-periodic, quasiperiodic and almost periodic motions, when the external fluctuation f in the atmospheric energy balance model is time-periodic, quasiperiodic and almost periodic, respectively.*

This result may be relevant to the El Nino-Southern Oscillation phenomenon. Ei Nino is a well-known climate phenomenon in the atmosphere-ocean system. Originally, it refers to a seasonal invasion, along the coast of Peru around Christmas, of a warm southward ocean current that displaced the north-flowing cold current. Today, it is regarded as a part of a phenomenon called El Nino-Southern Oscillation (ENSO), a continual but "quasi"-periodic, perhaps irregular, cycle of shifts in ocean and atmosphere condition that affect the globe climate [24, 9, 29].

6 Summary

We have investigated the dynamical behavior of a coupled atmosphere-ocean system.
First, we show that the coupled atmosphere-ocean system is stable under the external fluctuation in the atmospheric energy balance relation (Theorem 3.2). Then, we estimate the atmospheric temperature feedback in terms of the freshwater flux, heat flux and the external fluctuation at the air-sea interface, as well as the earth's longwave radiation and the shortwave solar radiation (Theorem 4.1). Finally, we prove that the coupled atmosphere-ocean system has time-periodic, quasiperiodic and almost periodic motions (under suitable conditions on the physical quantities such as freshwater flux, the earth's longwave radiative cooling coefficient and the shortwave solar radiation profile), when the external fluctuation in the atmospheric energy balance relation is time-periodic, quasiperiodic and almost periodic, respectively (Theorem 5.4).

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